Discrete evolution operator for q-deformed top and Faddeev's modular double

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Plan

Notion of Faddeev's modular double

Faddeev's example. Consider the standard Heisenberg algebra (HA) generated by operators x, p

$$[x, p] = i .$$

Introduce the algebra T (quantum torus or Weyl pair) with generators U, V

$$U = e^{i\alpha x} , \quad V = e^{i\beta p} ,$$

(α, β are parameters) with commutation relations

$$U V = q V U \quad q = e^{-i\alpha\beta} .$$

One can think that T can be obtained from HA by means of log - function. So we have got a question:

Is the algebra T of quantum torus (in above realization) is "equivalent" (representation theories are identical) to the Heisenberg algebra?

<center>The answer is NO!</center>

Notion of Faddeev's modular double

Indeed, from HA one can construct another "dual" algebra $\widetilde{T}$ of quantum torus

$$\tilde{U} = e^{i\tilde{\alpha}x} \,, \quad \tilde{V} = e^{i\tilde{\beta}p} \,.$$

$$\tilde{U}\,\tilde{V} = \tilde{q}\,\tilde{V}\,\tilde{U} \,, \quad \tilde{q} = e^{-i\,\tilde{\alpha}\tilde{\beta}} \,,$$

with another parameters $\tilde{\alpha}, \tilde{\beta}$. Then, if $\boxed{\tilde{\alpha} = \frac{2\pi}{\beta} \,, \quad \tilde{\beta} = -\frac{2\pi}{\alpha} \,,}$, the

generators U, V of T commute with $\tilde{U}\,\tilde{V}$ of $\widetilde{T}$ and parameters q and $\tilde{q}$ are related by modular transformation

$$q = e^{-i\,\alpha\beta} = e^{i2\pi\tau} \;\rightarrow\; \tilde{q} = e^{-i\,\tilde{\alpha}\tilde{\beta}} = e^{-\frac{i2\pi}{\tau}} \quad (\tau \;\rightarrow\; \tilde{\tau} = -\frac{1}{\tau}) \,.$$

The double of algebras T and $\widetilde{T}$ is called modular double.
The modular double of T and $\tilde{T}$ is "equivalent to HA"!
The notion of the modular double was introduced by L.D.Faddeev in his fundamental paper in 1999.

Let x be a coordinate and p be a momentum of a free particle. The time evolution is defined by the evolution operator $\Theta(t) = \exp(\frac{i}{2}p^2 t)$:

$$p \to \Theta(t) \cdot p \cdot \Theta(t)^{-1} = p, \quad x \to \Theta(t) \cdot x \cdot \Theta(t)^{-1} = x + p\,t.$$

For coordinates U, V of quantum torus T, we obtain the evolution

$$V \to \Theta(t) \cdot V \cdot \Theta(t)^{-1} = V, \quad U \to \Theta(t) \cdot U \cdot \Theta(t)^{-1} = U\,e^{i\alpha tp}\,e^{\frac{i\alpha^2 t}{2}},$$

and for special interval $t = \frac{\beta}{\alpha} = -\frac{\tilde{\beta}}{\tilde{\alpha}}$ we obtain intrinsic <u>discrete</u> evolution on T

$$V \to \Theta \cdot V \cdot \Theta^{-1} = V, \quad U \to \Theta \cdot U \cdot \Theta^{-1} = U V q^{-\frac{1}{2}}, \tag{1}$$

where we denote $\Theta = \Theta(\frac{\beta}{\alpha})$.

Let's try to solve eqs. (1) from the beginning. The first eq. in (1) is $[V, \Theta] = 0$ and we search Θ as $\theta(V)$. The second eq. in (1) gives:

$$\theta(V) = q^{\frac{1}{2}}\,\theta(qV)\,V,$$

which can be solved in terms of the Jacobi theta-function ($|q| < 1$)

$$\theta(V) = \prod_{n=1}^{\infty}(1 + q^{n-1/2}V)\prod_{n=1}^{\infty}(1 + q^{n-1/2}V^{-1}).$$

The discret evolution operator $\Theta = \exp(\frac{i}{2}p^2\,t)\big|_{t=\frac{\alpha}{\beta}}$ should be proportional to the "compact" evolution operator $\theta(V, q)$:

$$\Theta = \exp(\tfrac{i}{2}p^2\,t)\Big|_{t=\frac{\alpha}{\beta}} \sim C(\tilde{V}, \tilde{U}) \cdot \theta(V, q)\,.$$

Recall that $\frac{\tilde{\beta}}{\tilde{\alpha}} = -\frac{\beta}{\alpha} = -t$ and it means that discrete evolution for $\tilde{V}, \tilde{U}$ is defined by Θ^{-1}: $\tilde{V} \to \Theta^{-1} \cdot \tilde{V} \cdot \Theta = \tilde{V}\,, \quad \tilde{U} \to \Theta^{-1} \cdot \tilde{U} \cdot \Theta = \tilde{U}\,\tilde{V}\,\tilde{q}^{-\frac{1}{2}}$.
Thus, the solution for Θ^{-1} is given as before

$$\Theta^{-1} \sim \theta(\tilde{V}, \tilde{q}) = \prod_{n=1}^{\infty}(1 + \tilde{q}^{n-1/2}\,\tilde{V}) \prod_{n=1}^{\infty}(1 + \tilde{q}^{n-1/2}\,\tilde{V}^{-1})\,,$$

which is "compact" evolution operator for dual quantum torus $\tilde{\mathcal{T}}$.
Finally the answer for Θ is given by known identity for theta-functions

$$\exp\left(\frac{i}{2}p^2\,\frac{\beta}{\alpha}\right) \sim \frac{\Theta(V, q)}{\Theta(\tilde{V}, \tilde{q})}\,.$$

Below we obtain these formulas in the context of a discrete evolution of $SL_q(2)$- quantum top considered by Faddeev and Alekseev.

As the analog of Weyl pair U, V (quantum torus) we will consider the "Heisenberg double" of the RTT algebra and the $RLRL$ - or reflection equation algebra.

1. *R*-matrices

Let *V* be a finite dimensional **C**-linear space. For any operator $X \in \mathrm{End}(V \otimes V)$ and integers $i > 0, j > 0$ we denote

$$X_{i\,i+1} := I^{\otimes(i-1)} \otimes X \otimes I^{\otimes(j-1)} \in \mathrm{End}(V^{\otimes(i+j)}),$$

where $I \in \mathrm{Aut}(V)$ is the identity operator.

Def 1. An operator $\hat{R} \in \mathrm{Aut}(V \otimes V)$ is called an R-*matrix* if

$$\boxed{\hat{R}_{12}\,\hat{R}_{23}\,\hat{R}_{12} \;=\; \hat{R}_{23}\,\hat{R}_{12}\,\hat{R}_{23} \in \mathrm{Aut}(V \otimes V \otimes V)\,.}$$

Def 2. An R-matrix $\hat{R}$ is called a Hecke type R-matrix if

$$\boxed{(\hat{R} - q\,\mathbf{1})(\hat{R} + q^{-1}\,\mathbf{1}) = 0\,,} \quad (\mathbf{1} = I \otimes I)\,.$$

1. *R*-matrices

Consider the set of antisymmetrizers $\mathcal{A}^{(k)}(q)$ which can be defined by recurrent relations: $\mathcal{A}^{(1)} = 1$,

$$\mathcal{A}^{(k+1)} = \frac{[k]_q}{[k+1]_q} \mathcal{A}^{(k)} \left(\frac{q^k}{[k]_q} - \hat{R}_k \right) \mathcal{A}^{(k)} \in \mathrm{End}(V^{\otimes(k+1)}).$$

Def 3. A Hecke type R-matrix $\hat{R}$ for $\underline{q - \text{generic}}$ is called $\underline{GL_q(n) \text{ type}}$ $\underline{\text{R-matrix}}$ if it satisfies

1.) $\mathcal{A}^{(n+1)} = 0 \;\Leftrightarrow\; \mathcal{A}^{(n)} \Big(\frac{q^n}{[n]_q} I - \hat{R}_n \Big) \mathcal{A}^{(n)} = 0$, 2.) $\mathrm{rk}(\mathcal{A}^{(n)}) = 1$.

An example of $GL_q(n)$ type R-matrix is the standard Drinfeld-Jimbo's R-matrix

$$\hat{R}^\circ = \sum_{i,j=1}^{n} q^{\delta_{ij}} E_{ij} \otimes E_{ji} + (q - q^{-1}) \sum_{i<j} E_{ii} \otimes E_{jj},$$

where $(E_{ij})_{kl} := \delta_{ik}\delta_{jl}$ are $(n \times n)$ matrix units.

Def 4. $\hat{R}$ is called *skew invertible* if $\exists\, \Psi \in End(V^{\otimes 2})$ such that

$$\hat{R}^{i_1 m_2}_{j_1 k_2}\, \Psi^{k_2 i_3}_{m_2 j_3} = \Psi^{i_1 m_2}_{j_1 k_2}\, \hat{R}^{k_2 i_3}_{m_2 j_3} = \delta^{i_1}_{j_3}\, \delta^{i_3}_{j_1}\,.$$

With any skew invertible $\hat{R}$ we associate matrix $D \in End(V)$:

$$D_1 = \mathrm{Tr}_{(2)}\Psi_{12}\,,$$

where $\mathrm{Tr}_{(i)}$ – trace in i-th space. Then, we define a quantum trace
(*q*-traces) for any quantum matrix Y

$$Y \mapsto \mathrm{Tr}_D(Y) := \mathrm{Tr}(D\,Y)\,,$$

which possesses many remarkable properties, e.g.,

$$\mathrm{Tr}_{D(2)}(\hat{R}^{\varepsilon}_{12}\, Y_1\, \hat{R}^{-\varepsilon}_{12}) = I_1\,\mathrm{Tr}_D(Y)\ \ (\varepsilon = \pm 1)\,,$$

$$\mathrm{Tr}_{D(1,\ldots,k)}\left(\left[\hat{R}_{i\,i+1},\ Y_{(1\ldots k)}\right]\right) = 0\ \ (\forall\, 1 < i < k\,,\ \forall Y_{(1\ldots k)})\,.$$

3. *RTT* and Reflection equation (RE) algebras

Quantized functions over matrix group (RTT algebra)
(L.Faddeev,N.Reshetikhin,L.Takhtajan (1989)).

Let $\hat{R}$ be a skew invertible R-matrix. Consider an associative unital
algebra generated by matrix components $\|T_j^i\|_{i,j=1}^{\dim V}$ which satisfy

$$\boxed{\hat{R}_{12}\, T_1\, T_2 = T_1\, T_2\, \hat{R}_{12}\,.}$$

The extension of this algebra by a set of components $\|(T^{-1})_j^i\|_{i,j=1}^{\dim V}$:

$$\sum_k T_k^i\,(T^{-1})_j^k = \sum_k (T^{-1})_k^i\, T_j^k = \delta_j^i\, \mathbf{1}\,,$$

is a Hopf algebra with coproduct, counit and antipode mappings:

$$\Delta(T_j^i) = \sum_k T_k^i \otimes T_j^k\,, \qquad \epsilon(T_j^i) = \delta_j^i\,, \qquad S(T_j^i) = (T^{-1})_j^i\,.$$

This algebra is called an *RTT algebra* and denoted by $\mathcal{F}[\hat{R}]$.

Def 5. Let $\hat{R}$ be a skew invertible R-matrix. An associative unital algebra $\mathcal{L}[\hat{R}]$ with generators $\|L_j^i\|_{i,j=1}^{\dim V}$ satisfying relations

$$L_1\,\hat{R}_{12}\,L_1\,\hat{R}_{12}\;=\;\hat{R}_{12}\,L_1\,\hat{R}_{12}\,L_1\,,$$

is called a _reflection equation (RE) algebra_.

Consider REA $\mathcal{L}[\hat{R}]$ for Hecke type $\hat{R}$ and introduce elements ($a_0 = 1$)

$$a_i = \mathrm{Tr}_{D(1,\dots,i)}\left(\mathcal{A}^{(i)} L_{\overline{1}}\dots L_{\overline{i}}\right),\quad p_i = \mathrm{Tr}_D(L^i)\ \ (i \geq 1)$$

where $L_{\overline{1}} := L_1\,,\quad L_{\overline{k+1}} := \hat{R}_k\,L_{\overline{k}}\,\hat{R}_k^{-1}$. Elements p_i and a_i are central and called _power sums_ and _elementary symmetric functions_, resp.
Prop. 1. Quantum Newton relations and q- Cayley-Hamilton identity

$$i_q\,a_i + (-1)^i \sum_{j=0}^{i-1}(-q)^j a_j\,p_{i-j} = 0 \quad \forall\,i \geq 1\,,$$
$$\sum_{j=0}^n (-q)^j\,a_j\,L^{n-j}\;=\;0\,.$$

Prop. 2 *The set of* elementary symmetric functions $\{a_j, \; j = 1, ..., n\}$ *generate the whole center in REA* $\mathcal{L}[\hat{R}_{GL_q(n)}]$.

> **Def 6.** A spectral extension of REA $\mathcal{L}[\hat{R}]$ for $GL_q(n)$ type $\hat{R}$-matrix is the extension of $\mathcal{L}[\hat{R}]$ by a set of invertible central elements μ_α ($\alpha = 1, \ldots, n$) such that
> $$[\mu_\alpha, L_j^i] = 0$$
> and $a_i = \sum_{1 \leq j_1 < \cdots < j_i \leq n} \mu_{j_1} \mu_{j_2} \ldots \mu_{j_i} \quad \forall \, i = 1, \ldots, n.$

It means that the Cayley-Hamilton identity can be written in factorized form

$$\sum_{j=0}^{n} (-q)^j \, a_j \, L^{n-j} = \prod_{\alpha=1}^{n} (L - q\mu_\alpha I) = 0 \, .$$

We need projectors

$$P^\alpha = \prod_{\beta \neq \alpha} \frac{(q^{-1}L - \mu_\beta)}{(\mu_\alpha - \mu_\beta)}$$

4. Heisenberg double of *RTT* and RE algebras

Def 6. A *Heisenberg double (HD) algebra* of the *RTT* and RE algebras is an associative unital algebra generated by elements $T_j^i \in \mathcal{F}[\hat{R}]$ and $L_j^i \in \mathcal{L}[\hat{R}]$ subject to commutation relations

$$\hat{R}_{12} \, T_1 \, T_2 = T_1 \, T_2 \, \hat{R}_{12} \, .$$

$$L_1 \, \hat{R}_{12} \, L_1 \, \hat{R}_{12} \, = \, \hat{R}_{12} \, L_1 \, \hat{R}_{12} \, L_1 \, ,$$

$$\gamma^2 \, T_1 \, L_2 \, = \, \hat{R}_{12} \, L_1 \, \hat{R}_{12} \, T_1 \, , \quad (\gamma \in \{\mathbf{C}\backslash 0\}) \, .$$

This algebra is a quantization of the Poisson structure on $T^*(GL(n))$:

$$\{T_j^i, \, T_m^k\} = 0, \ \{\ell_j^i, \, \ell_m^k\} = 2(\delta_m^i \ell_j^k - \delta_j^k \ell_m^i), \ \{\ell_j^i, \, T_m^k\} = \delta_j^k T_m^i.$$

4. Heisenberg double of *RTT* and RE algebras

$$T_j^i \to T_j^i \,, \quad L_j^i \to \delta_j^i + h\,\ell_j^i + \dots \,, \quad R_{km}^{ij} \to \delta_m^i \delta_k^j + h\,\delta_k^i \delta_m^j + \dots \,.$$

HD algebra is interpreted as quantum group cotangent bundle, where *RTT* algebra is a base and RE algebra is a bundle.

For the spectral extension of HD we have additional commutators of T_k^i and L_k^i with spectral elements $\{\mu_\alpha\}$

$$[\mu_\alpha, L_k^i] = 0, \quad \mu_\alpha\, T_k^i - \gamma^2 T_k^i\, \mu_\alpha = (1 - q^2)\mu_\alpha(P^\alpha\, T)_k^i \,.$$

5. Discrete time evolution on quantum group cotangent bundle

Consider sequence of automorphisms on the HD $(\mathcal{F} \sharp \mathcal{L})[\hat{R}]$

$$\{T, \ L\} \xrightarrow{\ \theta^k\ } \{T(k), \ L(k)\}, \ \ \forall \, k = 0, 1, 2, \ldots,$$

$$\hat{R}_{12} \, T_1(k) \, T_2(k) \ = \ T_1(k) \, T_2(k) \hat{R}_{12}$$

$$\hat{R}_{12} \, L_1(k) \hat{R}_{12} \, L_1(k) \ = \ L_1(k) \, \hat{R}_{12} \, L_1(k) \hat{R}_{12} \,,$$

$$\gamma^2 \, T_1(k) \, L_2(k) \ = \ \hat{R}_{12} \, L_1(k) \, \hat{R}_{12} \, T_1(k) \,.$$

Here k is a discrete time. For any $\hat{R}$-matrix these automorphisms can be realized as (Faddeev–Alekseev discrete time evolution for the quantum top)

$$T(k) = L^k \, T \,, \quad L(k) = L \,.$$

One can compare with: $U^{(k)} = (q^{\frac{1}{2}} V)^k \, U \,, \quad V^{(k)} = V.$

5. Discrete time evolution for $SL_q(n)$ case

Consider the case when RTT algebra is $SL_q(n)$ quantum group. In this case we require

$$\det_q(T) = \mathrm{Tr}_{(1,\ldots,n)} \left(\mathcal{A}^{(n)} T_1 \, T_2 \cdots T_n \right) = 1 \ .$$

Discrete time evolution must conserve this relation, i.e., we have $\det_q(L^k \, T) = 1 \ (\forall k > 0)$. This leads to the conditions

$$a_n = \mathrm{Tr}_{D(1,\ldots,n)} \left(\mathcal{A}^{(n)} L_{\overline{1}} L_{\overline{2}} \cdots L_{\overline{n}} \right) = q^{-1} \ , \quad \gamma^n = q \ .$$

We will investigate the discrete evolution for HD of $SL_q(N)$ type. The key point is that $\exists$ the special evolution operator Θ:

$$T(k+1) = L \, T(k) = \Theta \, T(k) \, \Theta^{-1} \ , \quad L(k+1) = \Theta \, L(k) \, \Theta^{-1} = L \ .$$

For the case of "ribbon Hopf algebra" the Faddeev-Alekseev evolution is given by $\Theta = $ ribbon element.

6. Evolution operator Θ for $SL_q(n)$ case.

Thus, we have for the first shift $k = 1$:

$$L\,T = \Theta\,T\,\Theta^{-1}\,, \quad L = \Theta\,L\,\Theta^{-1}\,, \tag{2}$$

and we assume $\Theta = \Theta(\mu_1, \ldots, \mu_n)$, where $\prod_{\alpha=1}^{n} \mu_\alpha = q^{-1}$.

For the HD with $\hat{R}$-matrix of the $SL_q(n)$-type the evolution operator $\Theta(\mu_\alpha)$ is a solution of first eq. in (2) which is written as

$$\Theta\big(\nabla^\alpha(\mu_\beta)\big) = q^{-1}\mu_\alpha^{-1}\,\Theta(\mu_\beta) \quad \forall\,\alpha = 1, \ldots, n\,, \tag{3}$$

where ∇^α are finite shift operators $\nabla^\alpha(\mu_\beta) := q^{2X_{\alpha\beta}}\,\mu_\beta$ and the matrix X is a Gram matrix

$$X_{\alpha\beta} = \langle \vec{e}_\alpha^{\,*}, \vec{e}_\beta^{\,*}\rangle = \delta_{\alpha\beta} - \frac{1}{n} \quad (\alpha, \beta = 1, \ldots, n)\,,$$

for the set of vectors: $\vec{e}_\alpha^{\,*} = \frac{1}{n}\,(\,\underbrace{-1, \ldots, -1}_{(\alpha-1)\ \text{times}}, n-1, -1, \ldots, -1\,)$.

As a result we obtain (special solution):

Proposition. *In case $|q| < 1$ a solution is expressed via multidimensional theta-function*

$$\Theta^{(1)}(\mu_\alpha) = \theta(\vec{p}, \Omega) = \sum_{\vec{k} \in \mathbf{Z}^{n-1}} \exp\left\{ \pi i \, (\vec{k}, \, \Omega \, \vec{k}) + 2\pi i \, (\vec{k}, \, \vec{p}) \right\},$$

where τ is a modular parameter, Ω is $(n-1) \times (n-1)$ matrix of periods

$$q = \exp(2\pi i \, \tau), \quad q^{1/n}\mu_\alpha = \exp(2\pi i \, p_\alpha), \quad \sum_{\alpha=1}^{n} p_\alpha = 0,$$

$$\Omega_{\alpha\beta} = \tfrac{2\tau}{n} A^*_{\alpha\beta} = 2\tau \left(\delta_{\alpha\beta} - \tfrac{1}{n} \right),$$

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Expression $\Theta^{(1)}(\mu_\alpha)$ converges either if $|q| < 1$, or if $q^m = 1$ (the series is truncated).

The $(n-1) \times (n-1)$ matrix $A^*_{\alpha\beta}$ is a Gram matrix of a lattice A^*_{n-1} dual to the root lattice $A_{n-1} = sl(n)$, since we have $A^{*\,-1}_{\alpha\beta} = A_{\alpha\beta} = (\delta_{\alpha\beta} + 1)$ and $A_{\alpha\beta} = (e\alpha, e_\beta)$, where vectors $e_\alpha = (\underbrace{0, \ldots, 0}_{(\alpha-1) \text{ times}}, 1, 0, \ldots, 0, -1)$

form the basis in the root space of $sl(n)$.

7. "Noncompact" solution for the evolution operator Θ

Proposition. *In case $|q| \geq 1$ one can find another solution:*

$$\Theta^{(2)}(p_\alpha) := \exp\left(-\frac{\pi\mathrm{i}}{2\tau}\sum_{\beta=1}^{n} p_\beta^2\right),$$

of the evolution equations.

Written in the independent variables $\vec{p} = \{p_1, \ldots, p_{n-1}\}$ it reads

$$\Theta^{(2)}(\vec{p}) = \exp\left(-\frac{\pi\mathrm{i}}{\tau}\sum_{1\leq\alpha\leq\beta\leq n-1} p_\alpha p_\beta\right) = \exp\left\{-\pi\mathrm{i}\,(\vec{p},\,\Omega^{-1}\vec{p})\right\},$$

where the inverse matrix of periods is

$$\Omega^{-1}_{\alpha\beta} = \frac{1}{2\tau}\left(\delta_{\alpha\beta} + 1\right) = \frac{1}{2\tau}\,A_{\alpha\beta}\,,$$

and $A_{\alpha\beta} = \langle e_\alpha, e_\beta \rangle$ is the Gram matrix for the root lattice A_{n-1}. Note that the logarithmic change of variables: $\log(\mu_\alpha)/(2\pi i) = p_\alpha - \tau/n$ which was rather superficial in case of $\Theta^{(1)}$, is inevitable for the derivation of $\Theta^{(2)}$.

Finally, we comment on relation between the two evolution operators $\Theta^{(1)} = \theta(\vec{p}, \Omega)$ and $\Theta^{(2)}$. The relation is based on the identity for multidimensional theta functions

$$\theta(\Omega^{-1}\vec{p}, -\Omega^{-1}) = \left(\det(\Omega/\mathrm{i})\right)^{\frac{1}{2}} \exp\left\{\pi\mathrm{i}(\vec{p}, \Omega^{-1}\vec{p})\right\} \theta(\vec{p}, \Omega).$$

With our particular matrix of periods Ω we find

$$\Theta^{(2)}(\vec{p}) = \frac{1}{\sqrt{n}}\left(\frac{2\tau}{\mathrm{i}}\right)^{\frac{n-1}{2}} \frac{\theta(\vec{p}, \Omega)}{\theta(\Omega^{-1}\vec{p}, -\Omega^{-1})}.$$

Note that theta function $\theta(\Omega^{-1}\vec{p}, -\Omega^{-1})$ (in the denominator) commutes with the elements of HD (with $SL_q(n)$ $\hat{R}$-matrix) and can be thought as an evolution operator on a 'modular dual' quantum cotangent bundle associated to dual $\hat{R}$-matrix of $SL_{\tilde{q}}(n)$ type.

8. Example

In the $SL_q(2)$ case the evolution operator $\Theta^{(1)}$ becomes the Jacobi theta function (L.D. Faddeev (1995)):

$$\Theta^{(1)}(\mu_1) = \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}k(k+1)}\mu_1^k = \sum_{k\in\mathbb{Z}} \exp(\pi\mathrm{i}\,k^2\tau + 2\pi\mathrm{i}\,kz_1) = \theta_3(z_1; q)\,,$$

where $q = \exp(2\pi\mathrm{i}\,\tau)$, $\mu_1 = \exp(2\pi\mathrm{i}\,z_1)q^{-1/2}$. A multiplicative form for Θ is

$$\frac{1}{\eta(q)}\Theta^{(1)}(\mu_1) = \prod_{n=1}^{\infty}(1 + q^n\mu_1)(1 + q^{n-1}/\mu_1) = \prod_{n=1}^{\infty}(1 + q^n\sigma_1 + q^{2n-1})\,,$$

where $\eta(q) = \prod_{n=1}^{\infty}(1 - q^n)$. For dual evolution operator we have

$$\widetilde{\Theta}^{(1)}(\mu_1) = \sum_{k\in\mathbb{Z}} \exp(-\frac{\pi\mathrm{i}}{\tau}k^2 + \frac{2\pi\mathrm{i}}{\tau}kz_1) = \sum_{k\in\mathbb{Z}} \tilde{q}^{\frac{1}{2}k(k+1)}\tilde{\mu}_1^k\,,$$

where $\tilde{q} = \exp(-\frac{2\pi\mathrm{i}}{\tau})$, $\tilde{\mu}_1 = \exp(\frac{2\pi\mathrm{i}}{\tau}z_1)\tilde{q}^{-1/2}$.

- What is a dual HD for the standard HD of $SL_q(n)$ type (which centralize each other)?

- Explicit expressions for evolution operator Θ in the case of B, C, D quantum groups. In these cases Gram matrices A and their dual $A^* = (A)^{-1}$ are such that B and C type evolution operators are dual to each other.

- 3D analogue of RE (tetrahedron RE) were proposed in A.P.Isaev and P.P.Kulish, Mod. Phys. Lett. **A12** (1997) 427 (hep-th/9702013). The analog of 3D *RTT* algebra is also known: $R_{123}T_1T_2T_3 = T_3T_2T_1R_{123}$. What kind of cross-commutation relations are needed to describe discrete evolution in 3D case?